More Spanner Constructions

randomized spanner algorithm:

Input: $G=(V,E)$ (unweighted, undirected)

Output: $H=(V,F)$

$F \leftarrow \emptyset$

$C \leftarrow \emptyset$ "centers"

foreach $v \in V$

  add v to C with probability $p = \frac{1}{\sqrt{n}}$ (independent samples)

  Compute a BFS tree $T_v = (V, F_v)$ in G from v

  add edges of F_v to F

foreach $v \in V$

  if v has no neighbor in C

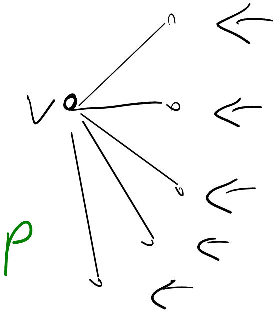
    $(*)$ add all edges $(v, v') \in E$ to F (all edges of v to its neighbors)

return $H=(V,F)$

Size of H

$\mathbb{E}[|C|] = p \cdot n = \sqrt{n}$ (expectation of binom. distribution with success prob. p)

edges of a BFS tree $T_v \leq n$ edges



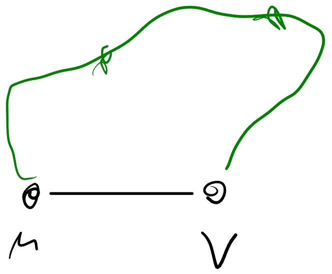
If v has no neighbors in C: $\mathbb{E}[\deg(v)] = \frac{1}{p} = \sqrt{n}$

expectation of geometric distribution with success prob. p

($\hat{=}$ # trials until first success)

$$\mathbb{E}[|F|] \leq \sqrt{n} \cdot n + n \cdot \sqrt{n} = O(n^{3/2})$$

Correctness (stretch 3)

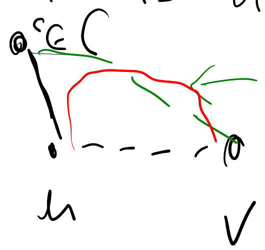


- Consider some edge $(u,v) \in E$
- If u has no neighbor in C

$\Rightarrow (u,v) \in F$ since v is a neighbor of u

$\Rightarrow \mathrm{dist}_H(u,v) = 1$

• If u has a neighbor c in C



v must be reachable in BFS tree of c

Let T_c be BFS tree of c in G

Triangle Inequality

$$\mathrm{dist}_H(u,v) \le \mathrm{dist}_{T_c}(u,v) \le \underbrace{\mathrm{dist}_{T_c}(u,c)}_{=1} + \underbrace{\mathrm{dist}_{T_c}(c,v)}_{=\mathrm{dist}_G(c,v)}$$

Triangle Inequality

$$\le 1 + \underbrace{\mathrm{dist}_G(c,u)}_{=1} + \underbrace{\mathrm{dist}_G(u,v)}_{=1} = 3$$

<u>Lemma</u> For all pairs $s,t \in V$: $\mathrm{dist}_H(s,t) \le \mathrm{dist}_G(s,t) + 2$
[Dor, Halperin, Zwick]

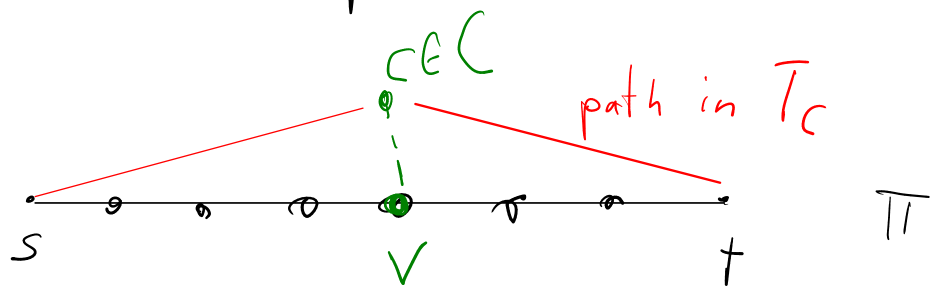
$$\le 3 \cdot \mathrm{dist}_G(s,t)$$

<u>Def</u>: A spanner H of G has <u>additive</u> stretch β if

$$\mathrm{dist}_H(s,t) \le \mathrm{dist}_G(s,t) + \beta \quad \text{for all pairs } s,t \in V$$

Proof of Lemma

Consider shortest path Π from s to t in G



Case 1: If no node on Π has a neighbor in C

$\Rightarrow$ All edges of Π are contained in H by spanner construction

$\Rightarrow \mathrm{dist}_H(s,t) \le |\Pi| = \mathrm{dist}_G(s,t)$

length of Π wrt ~~#~~edges

Case 2: If there is some node v on Π that has a neighbor $c \in C$

Let T_c be the BFS tree of c (in G)

$T_c \subseteq H$

$$\mathrm{dist}_H(s,t) \le \mathrm{dist}_H(s,c) + \mathrm{dist}_H(c,t) \le \mathrm{dist}_{T_c}(s,c) + \mathrm{dist}_{T_c}(c,t)$$

Δ-Inequality

$$= \mathrm{dist}_G(s,c) + \mathrm{dist}_G(c,t)$$

$$\le \mathrm{dist}_G(s,v) + \underbrace{\mathrm{dist}_G(v,c)}_{=1} + \underbrace{\mathrm{dist}_G(c,v)}_{=1} + \mathrm{dist}(v,t)$$

v on shortest path from s to t
$$= dist_G(s,v) + dist_G(v,t) + 2$$
$$\overset{\searrow}{=} dist_G(s,t) + 2 \qquad \square$$

Summary: +2-spanner with $O(n^{3/2})$ edges in expectation

Definition: An (α,β)-emulator $H=(V,F)$ of a graph $G=(V,E)$ is a weighted graph such that

$$dist_G(s,t) \underset{(1)}{\leq} dist_H(s,t) \underset{(2)}{\leq} \alpha \cdot dist_G(s,t) + \beta$$

for all pairs of nodes $s,t \in V$

+4-emulator construction

Input: $G=(V,E)$

Output $H=(V,F)$

$C \leftarrow \emptyset$

foreach $v \in V$: add v to C with probability $p = \frac{1}{n^{1/3}}$

foreach $v \in V$

if v has no neighbor in C:

size analysis

$E[|C|] = p \cdot n = n^{2/3}$
(binom. distr.)

$E[deg(v)] = \frac{1}{p} = n^{1/3}$
(geom. distr.)

add all edges $(v,v') \in E$ to F (of weight 1) $\rightarrow$ total $n \cdot n^{1/3}$ edges in exp.

otherwise

add an edge (v,c) to <u>one</u> neighbor $c \in C$ to F (of weight 1) $\rightarrow$ total $\leq n$

for each pair $x,y \in C$

add edge (x,y) of weight $w_H(x,y) = dist_G(x,y)$ to H $\rightarrow$ $E[|C|^2]$ many edges in total

Return $H = (V,F)$

<u>Theorem</u> H is a $+4$-emulator ($\beta = 4, \alpha = 1$) with $O(n^{4/3})$ edges in expectation.

<u>Proof</u>:

$$E[|C^2|] = (E[|C|])^2 + Var[|C|]$$

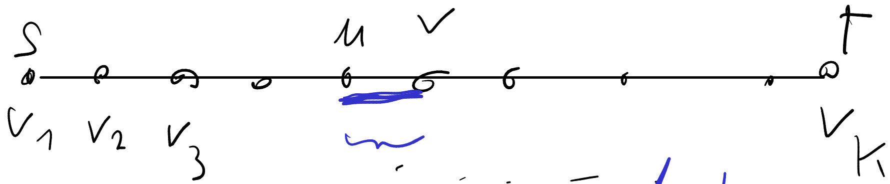
$$= (p \cdot n)^2 + p(1-p) \cdot n \leq p^2 n^2 + p \cdot n = n^{4/3} + n^{2/3} = O(n^{4/3})$$

$\Rightarrow$ In each step of the algorithm $O(n^{4/3})$ edges are added in total to H

$\Rightarrow$ H has $O(n^{4/3})$ many edges in expectation

Inequ. 1: observation: For all $(u,v) \in F$: $w_H(u,v) \geq dist_G(u,v)$ (by construction)

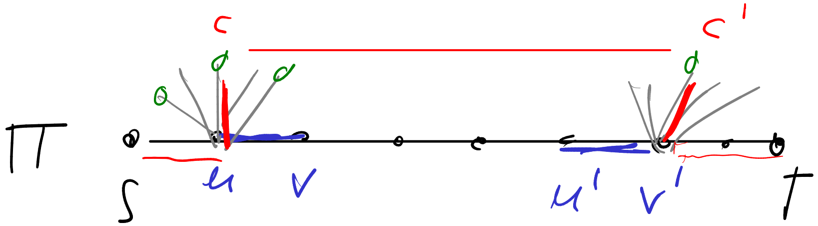
Let $s,t \in V$, let π be the shortest path from s to t in H



$w_H(u,v) \geq dist_G(u,v)$

$$\begin{aligned}
dist_G(s,t) &\overset{\Delta\text{-Inequ.}}{\leq} dist_G(\underset{s}{v_1}, v_2) + d_G(v_2, v_3) + \dots + dist_G(v_{k-1}, \underset{t}{v_k}) \\
&\leq w_H(v_1, v_2) + w_H(v_2, v_3) + \dots + w_H(v_{k-1}, v_k) \\
&= w_H(\pi) \\
&= dist_H(s,t) \checkmark
\end{aligned}$$

Inequ 2: Let $s,t \in V$, let π be the shortest path from s to t in G



Case 1: All edges of π contained in H

$$\Rightarrow \mathrm{dist}_H(s,t) \leq w_H(\pi) = w_G(\pi) = \mathrm{dist}_G(s,t)$$
$$\leq \mathrm{dist}_G(s,t) + 4$$

Case 2: Some edge of π not contained in H

Let (u,v) be the first edge on π not contained in H

Let (u',v') be the last ———— " ————

(It could also be the case that $(u,v) = (u',v')$)

Since $(u,v) \notin F$ and $(u',v') \notin F$:

both u and v' must have neighbors in C

(by construction of H)

$\Rightarrow (u,c) \in F$ for some $c \in C$ (by constr. of H)

$(v',c') \in F$ for some $c' \in C$

$$
\begin{aligned}
\mathrm{dist}_H(s,t) &\le \mathrm{dist}_H(s,u) + \overbrace{\mathrm{dist}_H(u,c)}^{=1} + \mathrm{dist}_H(c,c') + \\
&\qquad \underbrace{\mathrm{dist}_H(c',v')}_{=1} + \mathrm{dist}_H(v',t) \\
&= \mathrm{dist}_H(s,u) + \mathrm{dist}_H(c,c') + \mathrm{dist}_H(v',t) + 2 \\
&\le \mathrm{dist}_G(s,u) + \underbrace{\mathrm{dist}_H(c,c')}_{\substack{\le w_H(c,c') \\ = \mathrm{dist}_G(c,c') \\ \le \underbrace{\mathrm{dist}_G(c,u)}_{=1} + \mathrm{dist}_G(u,v') \\ + \underbrace{\mathrm{dist}_G(v',c')}_{=1}}} + \mathrm{dist}_G(v',t) + 2 \\
&\le \mathrm{dist}_G(s,u) + \mathrm{dist}_G(u,v') + \mathrm{dist}_G(v',t) + 4 \\
&= \mathrm{dist}_G(s,t) + 4 \qquad \square
\end{aligned}
$$